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STATIC ANALYSIS OF A CURVED SHAFT SUBJECTED TO END TORQUES

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Abstract—Both the exact closed-form solution and a numerical solution by the differential quadrature method (DQM) are obtained to predict the out-of-plane static behavior of a curved shaft subjected to end torques, based on the curved-beam versions of the classical (Bernoulli-Euler) and shear deformable (Bresse-Timoshenko) beam theories. Deflections, twist angles, bending moments, and twisting moments are calculated for the cases of a circular arc of circular cross section with clamped and simply supported boundary conditions, and results obtained by the two methods compared. The DQM gives good accuracy even when only a limited number of grid points is used.

1. INTRODUCTION

The out-of-plane behavior of a curved shaft due to torque has been previously treated by Eubanks (1963), Cheney (1965) and Bert (1989) based on the classical curved beam theory in which transverse shear deformation is not considered. The first paper was based on an unwieldy approach using the Kirchhoff rod equations, the second used thin-ring theory but obtained an erroneous solution, and the third was formulated from thin-ring theory and was solved directly assuming that the twisting moment was uniformly distributed. The purposes of the present work are: (1) to obtain exact solutions for out-of-plane deflections, twist angles, bending moments and twisting moments in a curved shaft due to end torques, based on both the classical and shear deformable beam theories; and (2) to demonstrate the application of the differential quadrature method to obtain accurate approximate solutions. Numerical results are presented for a circular arc of circular cross section with clamped and simply supported boundary conditions.

2. FORMULATION AND CLOSED-FORM SOLUTIONS

2.1. *Classical beam theory*

The curved shaft considered is shown in Fig. 1. The equilibrium equations for out-of-plane bending and twisting of a thin circular arc can be expressed as follows [cf. Volterra (1952)]:

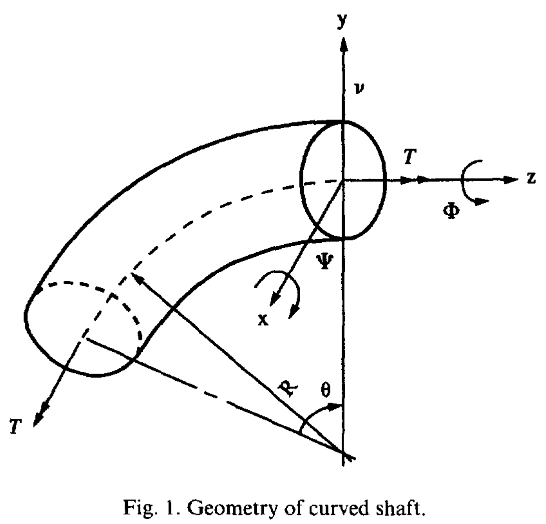

Fig. 1. Geometry of curved shaft.

$$M_x''+M_z'=0; \quad -M_x+M_z'=0 \tag{1}$$

where M_x and M_z are the respective bending and twisting moments at a given circumferential angular position $\bar{\theta}$, and a prime denotes differentiation with respect to $\bar{\theta}$.

The constitutive equations for small deflections and rotations are

$$M_x = (EI_x/R)(\Phi - R^{-1}v'') \tag{2}$$

$$M_z = (GJ/R)(\Phi' + R^{-1}v') \tag{3}$$

where EI_x and GJ are the respective flexural and torsional rigidities, R is the center-line radius of the member, v is the out-of-plane deflection, and Φ is the twist angle. Substituting M_x and M_z from eqns (2) and (3) into eqn (1) gives the following governing differential equations

$$-\frac{v''''}{R}+\frac{GJ}{EI_x}\frac{v''}{R}+\left(1+\frac{GJ}{EI_x}\right)\Phi''=0 \tag{4}$$

$$\left(1+\frac{GJ}{EI_x}\right)\frac{v''}{R}+\frac{GJ}{EI_x}\Phi''-\Phi=0. \tag{5}$$

Now, using v'' from eqn (5) in eqn (4), one obtains the following differential equation

$$\Phi''''+2\Phi''+\Phi=0 \tag{6}$$

which has the general solution

$$\Phi(\bar{\theta}) = C_1\cos\bar{\theta}+C_2\sin\bar{\theta}+C_3\bar{\theta}\sin\bar{\theta}+C_4\bar{\theta}\cos\bar{\theta} \tag{7}$$

where the C's are constants of integration. In view of eqn (5), the general solution for the out-of-plane deflection is

$$\frac{v(\bar{\theta})}{R} = -C_1\cos\bar{\theta}-C_2\sin\bar{\theta}-C_3\bar{\theta}\sin\bar{\theta}-C_4\bar{\theta}\cos\bar{\theta} - \frac{2C_3\cos\bar{\theta}}{(GJ/EI_x)+1}+\frac{2C_4\sin\bar{\theta}}{(GJ/EI_x)+1}+B_0\bar{\theta}+B_1 \tag{8}$$

where B_0 and B_1 are additional constants of integration.

Choosing the origin for $\bar{\theta}$ to be at the midpoint of the member and using the *anti-symmetric* nature of the problem, one may write

$$v(\bar{\theta}) = -v(-\bar{\theta}). \tag{9}$$

Thus, B_1, C_1, and C_3 must vanish.

If the member is simply supported flexurally at each end, then the boundary conditions can be expressed in the following form

$$M_x(\pm\alpha)=0, \quad v(\pm\alpha)=0, \quad M_z(\pm\alpha)=\pm T \tag{10}$$

where α is one half of the total included angle of the member and T is the applied torque at each end of the member. Thus

$$B_0 = \frac{TR}{GJ}, \quad C_2 = \frac{\alpha}{\sin\alpha}\left(\frac{TR}{GJ}\right), \quad C_4 = 0. \tag{11}$$

If the member is clamped flexurally at each end, then the boundary conditions can be expressed in the following form

$$v(\pm\alpha) = 0, \quad v'(\pm\alpha) = 0, \quad M_z(\pm\alpha) = \pm T. \tag{12}$$

Thus

$$B_0 = \frac{TR}{GJ}\frac{(GJ/EI_x)+1}{2q\cos\alpha+(GJ/EI_x)+1}, \quad C_4 = qB_0 \tag{13}$$

$$C_2 = \frac{1}{\sin\alpha}\left[C_4\left(\frac{2\sin\alpha}{(GJ/EI_x)+1} - \alpha\cos\alpha\right) + B_0\alpha\right] \tag{14}$$

where $q = (\sin\alpha - \alpha\cos\alpha)/(\sin\alpha\cos\alpha - \alpha)$.

2.2. *Shear-deformable beam theory*

Rao (1971), neglecting the warping deformation (as is appropriate for the circular cross section considered here), obtained the following equilibrium equations

$$\kappa\frac{G}{E}s\frac{v''}{R} - \kappa\frac{G}{E}s\Psi' = 0 \tag{15}$$

$$-\kappa\frac{G}{E}s\frac{v'}{R} + (1+\mu)\Phi' - \Psi'' + \left(\mu + \kappa\frac{G}{E}s\right)\Psi = 0 \tag{16}$$

$$\mu\Phi'' - \Phi + (1+\mu)\Psi' = 0 \tag{17}$$

where κ is the shear correction factor and Ψ is the angle of rotation due to pure out-of-plane bending. For simplicity of analysis, the following dimensionless variables have been introduced

$$s = AR^2/I_x, \quad \mu = GJ/EI_x \tag{18}$$

where A is the cross-sectional area, s is the slenderness ratio and μ is the rigidity ratio of the member. Substituting v' from eqn (16) into eqn (15) and using the *antisymmetric* nature of the problem gives the following general solutions

$$\Phi(\bar\theta) = C_6\sin\bar\theta + C_8\bar\theta\cos\bar\theta \tag{19}$$

$$\frac{v(\bar\theta)}{R} = -C_6\sin\bar\theta - C_8\bar\theta\cos\bar\theta + \frac{2C_8\sin\bar\theta}{\mu+1} + A_1\bar\theta \tag{20}$$

$$\Psi(\bar\theta) = -C_6\cos\bar\theta + C_8\left(\bar\theta\sin\bar\theta + \cos\bar\theta - \frac{2\mu}{\mu+1}\cos\bar\theta\right) + A_0. \tag{21}$$

If the member is simply supported flexurally at each end, then the boundary conditions can be expressed in the following form

$$M_x(\pm\alpha) = \pm\frac{EI_x}{R}(\Phi-\Psi') = 0, \quad v(\pm\alpha) = 0$$

$$M_z(\pm\alpha) = \pm\frac{GJ}{R}(\Psi+\Phi') = \pm T. \tag{22}$$

Thus

$$A_0 = \frac{TR}{GJ}, \quad A_1 = \left(1+\frac{J}{I_x\kappa s}\right)A_0 \tag{23}$$

$$C_6 = \frac{\alpha}{\sin\alpha}\left(1+\frac{J}{I_x\kappa s}\right)A_0, \quad C_8 = 0. \tag{24}$$

If the member is clamped flexurally at each end, then the boundary conditions can be expressed in the following form

$$\Psi(\pm\alpha) = 0, \quad v(\pm\alpha) = 0, \quad M_z(\pm\alpha) = \pm\frac{GJ}{R}(\Psi+\Phi') = \pm T. \tag{25}$$

Thus

$$A_0 = \frac{TR}{GJ}\frac{\mu+1}{2p\cos\alpha+\mu+1}, \quad A_1 = \left(1+\frac{J}{I_x\kappa s}\right)A_0 \tag{26}$$

$$C_6 = \frac{1}{\sin\alpha}\left[C_8\left(\frac{2\sin\alpha}{\mu+1}-\alpha\cos\alpha\right)+A_1\alpha\right], \quad C_8 = pA_0, \tag{27}$$

where $p = [\sin\alpha-(1+J/I_x\kappa s)\alpha\cos\alpha]/(\sin\alpha\cos\alpha-\alpha)$.

3. DIFFERENTIAL QUADRATURE METHOD

The differential quadrature method (DQM) was introduced by Bellman and Casti (1971). The method was applied for the first time in static beam analysis by Jang, Bert and Striz (1989). From a mathematical point of view, the application of the differential quadrature method to an ordinary differential equation can be expressed as follows:

$$L\{f(x)\}_i = \sum_{j=1}^{N} W_{ij}f(x_j) \tag{28}$$

where L denotes a differential operator, x_j are the discrete points considered in the domain, $f(x_j)$ are the function values at these points, W_{ij} are the weighting coefficients corresponding to the order of the derivative L and calculated based on the geometry of the discrete points x_i, and N denotes the number of discrete points in the domain. The general form of the function $f(x)$ is taken as

$$f(x) = x^{k-1} \quad \text{for} \quad k = 1,2,3,\ldots,N. \tag{29}$$

If the differential operator L represents an nth derivative, then

$$\sum_{j=1}^{N} W_{ij}x_j^{k-1} = (k-1)(k-2)\ldots(k-n)x_i^{k-n-1} \quad \text{for} \quad i,k = 1,2,\ldots,N. \tag{30}$$

This expression represents N sets of N linear algebraic equations which give a unique solution for the weighting coefficients, W_{ij}, since the coefficient matrix is a Vandermonde matrix which always has an inverse, as described by Hamming (1973). Thus, the method can be used to express the derivatives of a function at a discrete point in terms of the function values at all discrete points in the variable domain.

4. APPLICATION

The method of differential quadrature is applied here to the analysis of the out-of-plane behavior of a curved shaft based on the classical and shear-deformable beam theories. The differential quadrature approximations of the governing equations and boundary conditions are presented next.

4.1. *Classical beam theory*

Applying the differential quadrature method to eqns (4) and (5) gives

$$-\frac{1}{R\theta_0^4}\sum_{j=1}^{N} D_{ij}v_j+\frac{GJ}{EI_x}\frac{1}{R\theta_0^2}\sum_{j=1}^{N} B_{ij}v_j+\left(1+\frac{GJ}{EI_x}\right)\frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij}\Phi_j=0 \tag{31}$$

$$\left(1+\frac{GJ}{EI_x}\right)\frac{1}{R\theta_0^2}\sum_{j=1}^{N} B_{ij}v_j+\frac{GJ}{EI_x}\frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij}\Phi_j-\Phi_i=0 \tag{32}$$

where B_{ij} and D_{ij} are the weighting coefficients for the second and fourth derivatives, respectively, along the dimensionless axis X defined as

$$X=\theta/\theta_0. \tag{33}$$

Here θ is the circumferential angular position measured from the left support and $\theta_0(=2\alpha)$ is the total opening angle.

Considering the antisymmetry of the loading, one can express the boundary conditions for simply supported ends, given by eqn (10), and the deflection at the midpoint of the member in differential quadrature form as follows:

$$\text{at } X=0:\quad v_1=0$$

$$\text{at } X=0+\delta:\quad \frac{1}{\theta_0}\left(\sum_{j=1}^{N} A_{2j}\Phi_j+\frac{1}{R}\sum_{j=1}^{N} A_{2j}v_j\right)=\frac{TR}{GJ}$$

$$\text{at } X=0+\delta:\quad \frac{EI_x}{R}\left(\Phi_2-\frac{1}{R\theta_0^2}\sum_{j=1}^{N} B_{2j}v_j\right)=0$$

$$\text{at } X=0.5:\quad v_{(N+1)/2}=0 \tag{34}$$

$$\text{at } X=1-\delta:\quad \frac{EI_x}{R}\left(\Phi_{N-1}-\frac{1}{R\theta_0^2}\sum_{j=1}^{N} B_{(N-1)j}v_j\right)=0$$

$$\text{at } X=1-\delta:\quad \frac{1}{\theta_0}\left(\sum_{j=1}^{N} A_{(N-1)j}\Phi_j+\frac{1}{R}\sum_{j=1}^{N} A_{(N-1)j}v_j\right)=\frac{TR}{GJ}$$

$$\text{at } X=1:\quad v_N=0 \tag{35}$$

where A_{2j} and $A_{(N-1)j}$ are the weighting coefficients for the first derivatives. Here δ denotes a very small dimensionless distance measured along the dimensionless axis from each boundary end. This set of equations together with the appropriate boundary conditions can be solved for the deflections and twist angles.

Similarly, the boundary conditions for clamped ends, given by eqn (12), and the deflection at the midpoint of the member can be expressed in differential quadrature form as follows:

$$\text{at } X = 0: \quad v_1 = 0$$

$$\text{at } X = 0+\delta: \quad \frac{1}{\theta_0}\left(\sum_{j=1}^{N} A_{2j}\Phi_j + \frac{1}{R}\sum_{j=1}^{N} A_{2j}v_j\right) = \frac{TR}{GJ}$$

$$\text{at } X = 0+\delta: \sum_{j=1}^{N} A_{2j}v_j = 0$$

$$\text{at } X = 0.5: \quad v_{(N+1)/2} = 0 \tag{36}$$

$$\text{at } X = 1-\delta: \quad \sum_{j=1}^{N} A_{(N-1)j}v_j = 0$$

$$\text{at } X = 1-\delta: \quad \frac{1}{\theta_0}\left(\sum_{j=1}^{N} A_{(N-1)j}\Phi_j + \frac{1}{R}\sum_{j=1}^{N} A_{(N-1)j}v_j\right) = \frac{TR}{GJ}$$

$$\text{at } X = 1: v_N = 0. \tag{37}$$

The bending moments and twisting moments, given by eqns (2) and (3), can be expressed in differential quadrature form as follows:

$$M_x = \left(\frac{EI_x}{R}\right)\left(\Phi_i - \frac{1}{R\theta_0^2}\sum_{j=1}^{N} B_{ij}v_j\right) \tag{38}$$

$$M_z = \left(\frac{GJ}{R\theta_0}\right)\left(\sum_{j=1}^{N} A_{ij}\Phi_j + \frac{1}{R}\sum_{j=1}^{N} A_{ij}v_j\right). \tag{39}$$

4.2. *Shear-deformable beam theory*

Laura and Gutierrez (1993) applied the differential quadrature method to the analysis of vibrating Bresse-Timoshenko *straight beams*. Applying the method to shear-deformable curved beams, eqns (15)–(17), one obtains

$$\kappa\frac{G}{E}s\frac{1}{R\theta_0^2}\sum_{j=1}^{N} B_{ij}v_j - \kappa\frac{G}{E}s\frac{1}{\theta_0}\sum_{j=1}^{N} A_{ij}\Psi_j = 0$$

$$-\kappa\frac{G}{E}s\frac{1}{R\theta_0}\sum_{j=1}^{N} A_{ij}v_j + (1+\mu)\frac{1}{\theta_0}\sum_{j=1}^{N} A_{ij}\Phi_j - \frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij}\Psi_j + \left(\mu + \kappa\frac{G}{E}s\right)\Psi_i = 0$$

$$\mu\frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij}\Phi_j - \Phi_i + (1+\mu)\frac{1}{\theta_0}\sum_{j=1}^{N} A_{ij}\Psi_j = 0. \tag{40}$$

The boundary conditions for simply supported ends, given by eqn (22), and the deflection at the midpoint of the member can be expressed in differential quadrature form as follows:

$$\text{at } X = 0: \quad v_1 = 0$$

$$\text{at } X = 0+\delta: \quad \frac{1}{\theta_0}\sum_{j=1}^{N} A_{2j}\Phi_j + \Psi_2 = TR/GJ$$

$$\text{at } X = 0+\delta: \quad \frac{EI_x}{R}\left(\Phi_2 - \frac{1}{\theta_0}\sum_{j=1}^{N} A_{2j}\Psi_j\right) = 0$$

$$\text{at } X = 0.5: \quad v_{(N+1)/2} = 0 \tag{41}$$

$$\text{at } X = 1-\delta: \quad \frac{EI_x}{R}\left(\Phi_{N-1} - \frac{1}{\theta_0}\sum_{j=1}^{N} A_{(N-1)j}\Psi_j\right) = 0$$

$$\text{at } X = 1-\delta: \quad \frac{1}{\theta_0}\sum_{j=1}^{N} A_{(N-1)j}\Phi_j + \Psi_{(N-1)} = TR/GJ$$

$$\text{at } X = 1: \quad v_N = 0. \tag{42}$$

Similarly, the boundary conditions for clamped ends, given by eqn (25), and the deflection at the midpoint of the member can be expressed in differential quadrature form as follows:

$$\text{at } X = 0: \quad v_1 = 0$$

$$\text{at } X = 0+\delta: \quad \frac{1}{\theta_0}\sum_{j=1}^{N} A_{2j}\Phi_j + \Psi_2 = TR/GJ$$

$$\text{at } X = 0+\delta: \quad \Psi_2 = 0$$

$$\text{at } X = 0.5: \quad v_{(N+1)/2} = 0 \tag{43}$$

$$\text{at } X = 1-\delta: \quad \Psi_{N-1} = 0$$

$$\text{at } X = 1-\delta: \quad \frac{1}{\theta_0}\sum_{j=1}^{N} A_{(N-1)j}\Phi_j + \Psi_{(N-1)} = TR/GJ$$

$$\text{at } X = 1: \quad v_N = 0. \tag{44}$$

The bending moments and twisting moments, given by eqn (22), can be expressed in differential quadrature form as follows:

$$M_x = \left(\frac{EI_x}{R}\right)\left(\Phi_i - \frac{1}{\theta_0}\sum_{j=1}^{N} A_{ij}\Psi_j\right) \tag{45}$$

$$M_z = \left(\frac{GJ}{R}\right)\left(\Psi_i + \frac{1}{\theta_0}\sum_{j=1}^{N} A_{ij}\Phi_j\right). \tag{46}$$

5. NUMERICAL RESULTS AND COMPARISONS

Based on the above derivations, the deflections, twist angles, bending moments and twisting moments for the out-of-plane behavior of the member are calculated by a closed-form solution and by the differential quadrature method. The deflections, twist angles, bending moments and twisting moments are evaluated for the case of a circular arc with circular cross section under clamped and simply supported boundary conditions. Numerical results are compared between the two solution methods. The ratio of center-line radius R to radius of cross section r is 5.0, and the Poisson's ratio, v, of the member is 0.3.

For classical thin curved beam theory, Tables 1 and 2 present the results of convergence studies relative to the number of grid points N and the δ parameter, respectively. Table 1 shows that the accuracy of the numerical solution increases with increasing N, passes

Table 1. Twist angle $\Phi^* = \Phi GJ/TR$ for out-of-plane behavior of circular arc shaft and clamped ends with circular cross section for a range of grid points, using classical beam theory; $v = 0.3$, $\theta_0 = 180°$ and $\delta = 1\times10^{-5}$

		Number of grid points			
θ	Exact	7	9	11	13
0°	−0.8511	−0.8597	−0.8509	−0.8512	−0.8512

Table 2. Twist angle $\Phi^* = \Phi GJ/TR$ for out-of-plane behavior of circular arc shaft and clamped ends with circular cross section for a range of δ, using classical beam theory; $\nu = 0.3$, $\theta_0 = 180°$ and N = 13

		δ			
θ	Exact	1×10^{-3}	1×10^{-4}	1×10^{-5}	1×10^{-6}
0°	−0.8511	−0.8537	−0.8514	−0.8512	−0.8511

through a maximum, but then decreases due to numerical instabilities if N becomes too large. Table 2 shows how the numerical solution is sensitive to the choice of δ. The optimal value of δ is found to be 1×10^{-5} to 1×10^{-6}, which is obtained from trial-and-error calculations. The solution accuracy decreases due to numerical instabilities if δ becomes too small or too large. The remainder of the numerical results are computed with thirteen discrete points along the dimensionless x-axis and $\delta = 1\times10^{-5}$. For members with either simply supported or clamped ends and an opening angle of 180°, the results are summarized for classical beam theory in Tables 3 and 4. The shear correction factor κ is the established value (0.89) for a circular cross section using elasticity theory. Tables 5 and 6 summarize the results for such members with simply supported and clamped ends, respectively, and an opening angle of 180°. The bending moments and twisting moments are summarized in Tables 7–10. From Tables 3 and 5, for the case of clamped ends, the deflections v^* based on classical beam theory are larger than those based on shear-deformable beam theory, but the twist angles Φ^* are smaller. In Tables 4 and 6, for the case of simply supported ends, both deflections and twist angles based on classical beam theory are smaller than those based on shear-deformable beam theory. From Tables 7–10, the twisting moment distribution M_z^* is uniform for simply supported ends, but varies for clamped ends. The bending moment M_x^* is identically zero along the entire length of the member for simply supported ends, but varies for clamped ends. As can be seen, the numerical results show excellent agreement with the exact solutions.

Table 3. Deflection $v^* = vGJ/TR^2$ and twist angle $\Phi^* = \Phi GJ/TR$ for out-of-plane behavior of circular arc shaft and clamped ends with circular cross section, using classical beam theory; $\nu = 0.3$ and $\theta_0 = 180°$

	v^*		Φ^*	
θ	Exact	DQM	Exact	DQM
0°	0.0	0.0	−0.8511	−0.8512
18°	−0.009935	−0.009933	−0.5623	−0.5623
36°	−0.02435	−0.02435	−0.3359	−0.3359
54°	−0.02863	−0.02863	−0.1767	−0.1767
72°	−0.01897	−0.01897	−0.07281	−0.07281
90°	0.0	0.0	0.0	-2.0×10^{-7}

Table 4. Deflection $v^* = vGJ/TR^2$ and twist angle $\Phi^* = \Phi GJ/TR$ for out-of-plane behavior of circular arc shaft and simply supported ends with circular cross section, using classical beam theory; $\nu = 0.3$ and $\theta_0 = 180°$

	v^*		Φ^*	
θ	Exact	DQM	Exact	DQM
0°	0.0	0.0	−1.571	−1.571
18°	0.2373	0.2373	−1.494	−1.494
36°	0.3283	0.3283	−1.271	−1.271
54°	0.2950	0.2950	−0.9233	−0.9233
72°	0.1712	0.1712	−0.4854	−0.4854
90°	0.0	0.0	0.0	-5.5×10^{-7}

Table 5. Deflection $v^* = vGJ/TR^2$ and twist angle $\Phi^* = \Phi GJ/TR$ for out-of-plane behavior of circular arc shaft and clamped ends with circular cross section, using shear-deformable beam theory; $\nu = 0.3$, $\theta_0 = 180°$ and $R/r = 5.0$

	v^*		Φ^*	
θ	Exact	DQM	Exact	DQM
0°	0.0	0.0	−0.8864	−0.8865
18°	−0.004602	−0.004603	−0.5958	−0.5959
36°	−0.01697	−0.01697	−0.3645	−0.3645
54°	−0.02201	−0.02201	−0.1974	−0.1974
72°	−0.01512	−0.01512	−0.08371	−0.08371
90°	0.0	0.0	0.0	-2.1×10^{-7}

Table 6. Deflection $v^* = vGJ/TR^2$ and twist angle $\Phi^* = \Phi GJ/TR$ for out-of-plane behavior of circular arc shaft and simply supported ends with circular cross section, using shear-deformable beam theory; $\nu = 0.3$, $\theta_0 = 180°$ and $R/r = 5.0$

	v^*		Φ^*	
θ	Exact	DQM	Exact	DQM
0°	0.0	0.0	−1.606	−1.606
18°	0.2426	0.2426	−1.528	−1.528
36°	0.3357	0.3357	−1.300	−1.299
54°	0.3016	0.3016	−0.9440	−0.9440
72°	0.1751	0.1751	−0.4963	−0.4963
90°	0.0	0.0	0.0	-6.2×10^{-7}

Table 7. Bending moment $M_x^* = M_x/T$ and twisting moment $M_z^* = M_z/T$ for out-of-plane behavior of circular arc shaft and clamped ends with circular cross section, using classical beam theory; $\nu = 0.3$ and $\theta_0 = 180°$

	M_x^*		M_z^*	
θ	Exact	DQM	Exact	DQM
0°	−0.7197	−0.7197	1.0	1.0
18°	−0.6844	−0.6844	0.7776	0.7776
36°	−0.5822	−0.5822	0.5770	0.5770
54°	−0.4230	−0.4230	0.4178	0.4178
72°	−0.2224	−0.2224	0.3156	0.3156
90°	0.0	0.0	0.2803	0.2804

Table 8. Bending moment $M_x^* = M_x/T$ and twisting moment $M_z^* = M_z/T$ for out-of-plane behavior of circular arc shaft and simply supported ends with circular cross section, using classical beam theory; $\nu = 0.3$ and $\theta_0 = 180°$

	M_x^*		M_z^*	
θ	Exact	DQM	Exact	DQM
0°	0.0	2.6×10^{-9}	1.0	1.0
18°	0.0	2.4×10^{-6}	1.0	1.0
36°	0.0	2.5×10^{-6}	1.0	1.0
54°	0.0	2.5×10^{-6}	1.0	1.0
72°	0.0	2.2×10^{-6}	1.0	1.0
90°	0.0	1.6×10^{-6}	1.0	1.0

Table 9. Bending moment $M_x^* = M_x/T$ and twisting moment $M_z^* = M_z/T$ for out-of-plane behavior of circular arc shaft and clamped ends with circular cross section, using shear-deformable beam theory; $\nu = 0.3$, $\theta_0 = 180°$, and $R/r = 5.0$

	M_x^*		M_z^*	
θ	Exact	DQM	Exact	DQM
0°	−0.7197	−0.7197	1.0	1.0
18°	−0.6844	−0.6844	0.7776	0.7776
36°	−0.5822	−0.5822	0.5770	0.5770
54°	−0.4230	−0.4230	0.4178	0.4178
72°	−0.2224	−0.2224	0.3156	0.3156
90°	0.0	0.0	0.2803	0.2804

Table 10. Bending moment $M_x^* = M_x/T$ and twisting moment $M_z^* = M_z/T$ for out-of-plane behavior of circular arc shaft and simply supported ends with circular cross section, using shear-deformable beam theory; $\nu = 0.3$, $\theta_0 = 180°$ and $R/r = 5.0$

	M_x^*		M_z^*	
θ	Exact	DQM	Exact	DQM
0°	0.0	2.0×10^{-9}	1.0	1.0
18°	0.0	1.2×10^{-6}	1.0	1.0
36°	0.0	9.4×10^{-7}	1.0	1.0
54°	0.0	4.2×10^{-8}	1.0	1.0
72°	0.0	4.9×10^{-7}	1.0	1.0
90°	0.0	1.4×10^{-6}	1.0	1.0

The differential quadrature method can be applied in the case of a shaft of continuously varying cross section. The method was applied in the case of a beam of varying cross section by Laura and Gutierrez (1993).

6. CONCLUSIONS

Both closed-form analytical and differential quadrature methods were used to compute the deflections, twist angles, bending moments, and twisting moments for the out-of-plane behavior of a curved shaft under end torques, based on the classical and shear-deformable beam theories. The DQM gives results which agree very well with the exact ones for the cases treated while requiring only a limited number of grid points.

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